Calogero's 'goldfish' is indeed a school of free particles

2004 J. Phys. A: Math. Gen. 37 11391

(http://iopscience.iop.org/0305-4470/37/47/008)

J. Phys. A: Math. Gen. **37** (2004) 11391–11400 PII: S0305-4470(04)81582-8

Calogero's 'goldfish' is indeed a school of free particles

M C Nucci

Dipartimento di Matematica e Informatica, Università di Perugia, 06123 Perugia, Italy

Received 1 June 2004, in final form 4 October 2004
Published 10 November 2004
Online at stacks.iop.org/JPhysA/37/11391
doi:10.1088/0305-4470/37/47/008

Abstract

A many-body system of N nonlinear ordinary differential equations of second order which is amenable to exact treatments (a 'goldfish') (Calogero 2001 The neatest many-body problem amenable to exact treatments (a 'goldfish'?) *Physica* D **152–153** 78–84) is shown to be equivalent through an exact transformation to the equations of one-dimensional motion of $(N-1)$ free particles (a school of free particles, indeed). The transformation is obtained by applying the reduction method and Lie group analysis as introduced in Nucci (1996 The complete Kepler group can be derived by Lie group analysis *J. Math. Phys.* **37** 1772–5).

PACS numbers: 02.20.Sv, 45.50.Jf

1. Introduction

In [1] Calogero derived a solvable many-body problem, i.e.

$$\ddot{z}_n = 2\sum_{m=1, m\neq n}^{N} \frac{\dot{z}_n\dot{z}_m}{z_n - z_m} \qquad (n = 1, \ldots, N), \tag{1}$$

by considering the following solvable nonlinear partial differential equation:

$$\varphi_t + \varphi_x + \varphi^2 = 0, \qquad \varphi \equiv \varphi(x, t)$$

and looking at the behaviour of the poles of its solution. In [3] the same system (1) was presented, its properties were further studied and its solution was given in terms of the roots of the following algebraic equation in z:

$$\sum_{m=1}^{N} \frac{\dot{z}_m(0)}{[z - z_m(0)]} = \frac{1}{t}. \tag{2}$$

In that paper, Calogero called system (1) 'a goldfish' following a statement by Zakharov [21]. In the preface of [2], Calogero states: 'By "amenable to exact treatments" we mean that (...) significant progress can be made by "exact" (i.e., not approximate) techniques.'

He then introduces *three categories of problems—solvable, integrable, linearizable—(which) are ordered in terms of increasing difficulty*. In this paper we use the reduction method and Lie group analysis as introduced in [16] to show that system (1) is more than solvable[1] and integrable[2], it is actually linearizable, and can be transformed into the following trivial linear second-order system of $N-1$ equations

$$\frac{\mathrm{d}^2\tilde{u}_j}{\mathrm{d}\tilde{y}^2}=0 \qquad (j=1,\ldots,N-1), \tag{3}$$

which may be interpreted as the equations of one-dimensional motion of $N-1$ free particles, a school of free particles indeed.

Lie group analysis is the most powerful tool to find the general solution of ordinary differential equations. Any known integration technique[3] can be shown to be a particular case of a general integration method based on the derivation of the continuous group of symmetries admitted by the differential equation, i.e. the Lie symmetry algebra, which can easily be derived by a straightforward although lengthy procedure. As computer algebra software becomes widely used, the integration of systems of ordinary differential equations by means of Lie group analysis is becoming easier to perform. A major drawback of Lie's method is that it is useless when applied to systems of M first-order equations[4], because they admit an infinite number of symmetries, and there is no systematic way to find even a one-dimensional Lie symmetry algebra, apart from trivial groups such as translations in time admitted by autonomous systems. One may try to derive an admitted M-dimensional solvable Lie symmetry algebra by making an ansatz on the form of its generators.

However, in [16] we have remarked that any system of M first-order equations could be transformed into an equivalent system where at least one of the equations is of second order. Then the admitted Lie symmetry algebra is no longer infinite dimensional, and nontrivial symmetries of the original system could be retrieved [16]. This idea has been successfully applied in several instances [10, 16–18, 20]. Also in [12] we have shown that first integrals can be obtained by Lie group analysis even if the system under study does not come from a variational problem, i.e., without making use of Noether's theorem [13]. We remark that interactive (not automatic) programs for calculating Lie symmetries such as [14, 15] are most appropriate for performing this task.

In the next section we show in detail how to transform (1) into (3). In the third and last section we make some final remarks.

2. A school of free particles

Firstly we consider some particular values of N, i.e. $N = 2, 3, 4$, and then we consider general N.

[1] *Solvable models are characterized by the availability of a technique of solution* [2].
[2] *Integrable models are those for which some approach (for instance a 'Lax pair') is available* [2].
[3] We mean those taught in most undergraduate courses on ordinary differential equations.
[4] Any undergraduate science/engineering student knows that an M-order ordinary differential equation can be transformed into an equivalent system of M first-order equations. Less well known to students but common knowledge among experts in Lie group analysis is the dramatic consequence that that transformation has on the dimension of the admitted Lie symmetry algebra. In fact while the maximum Lie symmetry algebra admitted by a single M-order equation is finite [5], the dimension of the Lie symmetry algebra admitted by a system of M first-order equations is infinite.

2.1. Case $N = 2$

In this case system (1) reduces to

$$\ddot{z}_1 = 2\frac{\dot{z}_1\dot{z}_2}{z_1 - z_2}, \qquad \ddot{z}_2 = 2\frac{\dot{z}_2\dot{z}_1}{z_2 - z_1}. \tag{4}$$

If we introduce four new dependent variables w_1, w_2, w_3, w_4 such that

$$z_1 = w_1, \qquad z_2 = w_2, \qquad \dot{z}_1 = w_3, \qquad \dot{z}_2 = w_4 \tag{5}$$

then system (4) becomes an autonomous system of four first-order equations, i.e.

$$\dot{w}_1 = w_3, \qquad \dot{w}_2 = w_4, \qquad \dot{w}_3 = 2\frac{w_3 w_4}{w_1 - w_2}, \qquad \dot{w}_4 = 2\frac{w_4 w_3}{w_2 - w_1}. \tag{6}$$

If we follow the reduction method and introduce a new independent variable, say[5] $y = w_1$, then system (6) reduces to the following non-autonomous system of three equations of first order

$$w_2' = \frac{w_4}{w_3}, \qquad w_3' = 2\frac{w_3 w_4}{w_3(y - w_2)}, \qquad w_4' = 2\frac{w_4 w_3}{w_3(w_2 - y)} \tag{7}$$

where $'$ denotes derivative by y. If we derive w_4 from the first equation of system (7), i.e.

$$w_4 = w_2' w_3$$

then we obtain the following system of two equations, one of first order and one of second order

$$w_3' = \frac{-2w_2' w_3}{w_2 - y} \tag{8}$$

$$w_2'' = \frac{2w_2'(w_2' + 1)}{w_2 - y}. \tag{9}$$

It is noteworthy that equation (9) does not depend on w_3, and that equation (8) can easily be integrated as soon as the general solution of equation (9) is determined. Therefore, we can apply Lie group analysis to equation (9) only. Using the interactive REDUCE programs [14, 15], we obtain an eight-dimensional Lie symmetry algebra[6] generated by the following eight operators[7]

$$\begin{aligned}
\Gamma_1 &= \frac{uy}{u - y}(u^2\partial_u - y^2\partial_y), & \Gamma_2 &= \frac{uy}{u - y}(u\partial_u - y\partial_y) \\
\Gamma_3 &= \frac{uy}{u - y}(\partial_u - \partial_y), & \Gamma_4 &= \frac{1}{u - y}(y\partial_u - u\partial_y) \\
\Gamma_5 &= \frac{1}{u - y}(\partial_u - \partial_y), & \Gamma_6 &= u^2\partial_u + y^2\partial_y \\
\Gamma_7 &= u\partial_u + y\partial_y, & \Gamma_8 &= \partial_u + \partial_y
\end{aligned} \tag{10}$$

which means that equation (9), i.e.

$$u'' = \frac{2u'(u' + 1)}{u - y}, \tag{11}$$

is linearizable by means of a point transformation [11]. In order to find the linearizing transformation we have to look for a two-dimensional Abelian intransitive subalgebra [11],

[5] We could have chosen any other dependent variable as the new independent variable.
[6] This symmetry algebra is isomorphic to $sl(3, \mathbb{R})$ [5, 6].
[7] To simplify the notation we have replaced w_2 with u.

and, following Lie's classification of two-dimensional algebras in the real plane [11], we have to transform it into the canonical form

$$\partial_{\tilde{y}}, \qquad \tilde{u}\partial_{\tilde{y}}$$

with $\tilde{y}$ and $\tilde{u}$ the new independent and dependent variables, respectively. We found that one such subalgebra is that generated by Γ_2 and $\Gamma_4 + \Gamma_7$, i.e.

$$\Gamma_2 = \frac{uy}{u-y}(u\partial_u - y\partial_y), \qquad \Gamma_4 + \Gamma_7 = \frac{1}{u-y}(u\partial_u - y\partial_y). \tag{12}$$

Then, it is easy to derive that

$$\tilde{y} = y + u = z_1 + z_2, \qquad \tilde{u} = yu = z_1 z_2$$

and equation (11) becomes

$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}\tilde{y}^2} = 0 \tag{13}$$

which may be interpreted as the equation of one-dimensional motion of a single free particle.

2.2. *Case $N = 3$*

In this case system (1) reduces to

$$\begin{aligned}
\ddot{z}_1 &= 2\left(\frac{\dot{z}_1\dot{z}_2}{z_1 - z_2} + \frac{\dot{z}_1\dot{z}_3}{z_1 - z_3}\right),\\
\ddot{z}_2 &= 2\left(\frac{\dot{z}_2\dot{z}_1}{z_2 - z_1} + \frac{\dot{z}_2\dot{z}_3}{z_2 - z_3}\right),\\
\ddot{z}_3 &= 2\left(\frac{\dot{z}_3\dot{z}_1}{z_3 - z_1} + \frac{\dot{z}_3\dot{z}_2}{z_3 - z_2}\right).
\end{aligned} \tag{14}$$

If we introduce six new dependent variables $w_1, w_2, w_3, w_4, w_5, w_6$ such that

$$\begin{aligned}
z_1 &= w_1, \qquad z_2 = w_2, \qquad z_3 = w_3,\\
\dot{z}_1 &= w_4, \qquad \dot{z}_2 = w_5, \qquad \dot{z}_3 = w_6,
\end{aligned} \tag{15}$$

then system (14) becomes an autonomous system of six first-order equations, i.e.

$$\begin{aligned}
&\dot{w}_1 = w_4, \qquad \dot{w}_2 = w_5, \qquad \dot{w}_3 = w_6, \qquad \dot{w}_4 = 2\left(\frac{w_4 w_5}{w_1 - w_2} + \frac{w_4 w_6}{w_1 - w_3}\right)\\
&\dot{w}_5 = 2\left(\frac{w_5 w_4}{w_2 - w_1} + \frac{w_5 w_6}{w_2 - w_3}\right), \qquad \dot{w}_6 = 2\left(\frac{w_6 w_4}{w_3 - w_1} + \frac{w_6 w_5}{w_3 - w_2}\right).
\end{aligned} \tag{16}$$

If we follow the reduction method and introduce a new independent variable, say[8] $y = w_1$, then system (16) reduces to the following nonautonomous system of five equations of first order:

$$\begin{aligned}
&w_2' = \frac{w_5}{w_4}, \qquad w_3' = \frac{w_6}{w_4}, \qquad w_4' = 2\left(\frac{w_5}{y - w_2} + \frac{w_6}{y - w_3}\right)\\
&w_5' = 2\left(\frac{w_5}{w_2 - y} + \frac{w_5 w_6}{w_4(w_2 - w_3)}\right), \qquad w_6' = 2\left(\frac{w_6}{w_3 - y} + \frac{w_6 w_5}{w_4(w_3 - w_2)}\right)
\end{aligned} \tag{17}$$

where $'$ denotes derivative by y. If we derive w_5 from the first equation and w_6 from the second equation of system (17), i.e.

$$w_5 = w_2' w_4, \qquad w_6 = w_3' w_4$$

[8] We could have chosen any other dependent variable as the new independent variable.

then we obtain the following system of three equations, one of first order and two of second order

$$w_4' = 2w_4 \frac{w_2'(y - w_3) + w_3'(y - w_2)}{(w_2 - y)(w_3 - y)} \tag{18}$$

$$w_3'' = 2w_3' \frac{w_2'(w_3 - y)^2 + w_3'(w_2 - y)(w_3 - w_2) + (w_3 - w_2)(w_2 - y)}{(w_2 - y)(w_3 - y)(w_3 - w_2)} \tag{19}$$

$$w_2'' = 2w_2' \frac{w_2'(w_3 - y)(w_3 - w_2) - w_3'(w_2 - y)^2 + (w_3 - y)(w_3 - w_2)}{(w_2 - y)(w_3 - w_2)(w_3 - y)}. \tag{20}$$

It is noteworthy that the two second-order equations (19) and (20) do not depend on w_4, and that equation (18) can easily be integrated as soon as the general solution of system (19), (20) is determined. Therefore, we can apply Lie group analysis just to the system (19), (20), i.e.[9]

$$\begin{aligned} u_1'' &= 2u_1' \frac{u_2'(u_1 - y)^2 + u_1'(u_2 - y)(u_1 - u_2) + (u_1 - u_2)(u_2 - y)}{(u_2 - y)(u_1 - y)(u_1 - u_2)} \\ u_2'' &= 2u_2' \frac{u_2'(u_1 - y)(u_1 - u_2) - u_1'(u_2 - y)^2 + (u_1 - y)(u_1 - u_2)}{(u_2 - y)(u_1 - u_2)(u_1 - y)}. \end{aligned} \tag{21}$$

Using the interactive REDUCE programs [14, 15], we obtain a 15-dimensional Lie symmetry algebra[10] generated by the following 15 operators:

$$\Gamma_1 = \frac{u_1 u_2 y^4 \partial_y}{(y - u_1)(y - u_2)} + \frac{u_1^4 u_2 y \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} - \frac{u_1 u_2^4 y \partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\Gamma_2 = \frac{y^3(u_1 y + u_2 y - u_1 u_2)\partial_y}{(y - u_1)(y - u_2)} - \frac{(-u_1 y + u_2 y - u_1 u_2)u_1^3 \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} - \frac{u_2^3(u_2 y + u_1 u_2 - u_1 y)\partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\Gamma_3 = -\frac{u_1 u_2 y^3 \partial_y}{(y - u_1)(y - u_2)} - \frac{u_1^3 u_2 y \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} + \frac{u_1 u_2^3 y \partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\begin{aligned} \Gamma_4 = {} & \frac{y^2(-4u_2 y - 4u_1 y + 3y^2 + 8u_1 u_2)\partial_y}{(y - u_1)(y - u_2)} + \frac{(8u_2 y - 4u_1 y - 4u_1 u_2 + 3u_1^2)u_1^2 \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} \\ & + \frac{u_2^2(4u_2 y + 4u_1 u_2 - 3u_2^2 - 8u_1 y)\partial_{u_2}}{(-u_1 + u_2)(y - u_2)} \end{aligned}$$

$$\Gamma_5 = \frac{y(-u_1 u_2 + y^2)\partial_y}{(y - u_1)(y - u_2)} - \frac{(u_2 y - u_1^2)u_1 \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} + \frac{u_2(u_1 y - u_2^2)\partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\Gamma_6 = \frac{u_1 u_2 y^2 \partial_y}{(y - u_1)(y - u_2)} + \frac{u_1^2 u_2 y \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} - \frac{u_1 u_2^2 y \partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\Gamma_7 = y\partial_y + u_1 \partial_{u_1} + u_2 \partial_{u_2}$$

$$\Gamma_8 = -\frac{y(-u_2 - u_1 + 2y)\partial_y}{(y - u_1)(y - u_2)} + \frac{(y + u_2 - 2u_1)u_1 \partial_{u_1}}{(-u_1 + u_2)(y - u_1)} - \frac{u_2(y + u_1 - 2u_2)\partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

$$\Gamma_9 = \frac{(y - u_2 - u_1)\partial_y}{(y - u_1)(y - u_2)} - \frac{(y - u_1 + u_2)\partial_{u_1}}{(-u_1 + u_2)(y - u_1)} + \frac{(y - u_2 + u_1)\partial_{u_2}}{(-u_1 + u_2)(y - u_2)}$$

[9] To simplify the notation we have replaced w_2 with u_2, and w_3 with u_1.

[10] This symmetry algebra is isomorphic to $sl(4, \mathbb{R})$ [5, 6].

$$\Gamma_{10} = \frac{(-u_1u_2 - u_2y - u_1y + 3y^2)\partial_y}{(y-u_1)(y-u_2)} - \frac{(u_1y + u_2y + u_1u_2 - 3u_1^2)\partial_{u_1}}{(-u_1+u_2)(y-u_1)} + \frac{(u_1u_2 + u_2y + u_1y - 3u_2^2)\partial_{u_2}}{(-u_1+u_2)(y-u_2)}$$

$$\Gamma_{11} = y^2\partial_y + u_1^2\partial_{u_1} + u_2^2\partial_{u_2}$$

$$\Gamma_{12} = \frac{(u_1+u_2)\partial_y}{(y-u_1)(y-u_2)} + \frac{(u_2+y)\partial_{u_1}}{(-u_1+u_2)(y-u_1)} - \frac{(u_1+y)\partial_{u_2}}{(-u_1+u_2)(y-u_2)}$$

$$\Gamma_{13} = \frac{\partial_y}{(y-u_1)(y-u_2)} + \frac{\partial_{u_1}}{(-u_1+u_2)(y-u_1)} - \frac{\partial_{u_2}}{(-u_1+u_2)(y-u_2)}$$

$$\Gamma_{14} = \frac{u_1u_2\partial_y}{(y-u_1)(y-u_2)} + \frac{u_2y\partial_{u_1}}{(-u_1+u_2)(y-u_1)} - \frac{u_1y\partial_{u_2}}{(-u_1+u_2)(y-u_2)}$$

$$\Gamma_{15} = \frac{u_1u_2y\partial_y}{(y-u_1)(y-u_2)} + \frac{u_1u_2y\partial_{u_1}}{(-u_1+u_2)(y-u_1)} - \frac{u_1u_2y\partial_{u_2}}{(-u_1+u_2)(y-u_2)} \qquad (22)$$

which means that system (21) is linearizable [4, 19]. In order to find the linearizing transformation we look for a four-dimensional Abelian subalgebra $L_{4,2}$ of rank 1 and have to transform it into the canonical form [19]

$$\partial_{\tilde{y}}, \qquad \tilde{u}_1\partial_{\tilde{y}}, \qquad \tilde{u}_2\partial_{\tilde{y}}, \qquad \tilde{y}\partial_{\tilde{y}},$$

with $\tilde{y}$, $\tilde{u}_1$ and $\tilde{u}_2$ the new independent and dependent variables, respectively. We find that one such subalgebra is that generated by

$$\begin{aligned} X_1 &= \Gamma_{10} + \Gamma_{14} + \Gamma_8 = \frac{y^2\partial_y}{(y-u_1)(y-u_2)} + \frac{u_1^2\partial_{u_1}}{(u_1-u_2)(u_1-y)} + \frac{u_2^2\partial_{u_2}}{(u_2-u_1)(u_2-y)} \\ X_2 &= -\Gamma_4 + 3\Gamma_{11} + 6\Gamma_6 = (yu_1 + yu_2 + u_1u_2)X_1 \\ X_3 &= -\Gamma_3 = yu_1u_2X_1, \qquad X_4 = 2\Gamma_5 + \Gamma_7 + 3\Gamma_{15} = (y + u_1 + u_2)X_1. \end{aligned} \qquad (23)$$

Then, it is easy to derive that the linearizing transformation is

$$\begin{aligned} \tilde{y} &= y + u_1 + u_2 = z_1 + z_2 + z_3 \\ \tilde{u}_1 &= yu_1 + yu_2 + u_1u_2 = z_1z_2 + z_1z_3 + z_2z_3 \\ \tilde{u}_2 &= yu_1u_2 = z_1z_2z_3 \end{aligned} \qquad (24)$$

and system (21) becomes

$$\frac{\mathrm{d}^2\tilde{u}_1}{\mathrm{d}\tilde{y}^2} = 0, \qquad \frac{\mathrm{d}^2\tilde{u}_2}{\mathrm{d}\tilde{y}^2} = 0 \qquad (25)$$

which may be interpreted as the equations of one-dimensional motion of two free particles.

2.3. *Case* $N = 4$

In this case system (1) is

$$\begin{aligned} \ddot{z}_1 &= 2\left(\frac{\dot{z}_1\dot{z}_2}{z_1-z_2} + \frac{\dot{z}_1\dot{z}_3}{z_1-z_3} + \frac{\dot{z}_1\dot{z}_4}{z_1-z_4}\right), \qquad \ddot{z}_2 = 2\left(\frac{\dot{z}_2\dot{z}_1}{z_2-z_1} + \frac{\dot{z}_2\dot{z}_3}{z_2-z_3} + \frac{\dot{z}_2\dot{z}_4}{z_2-z_4}\right) \\ \ddot{z}_3 &= 2\left(\frac{\dot{z}_3\dot{z}_1}{z_3-z_1} + \frac{\dot{z}_3\dot{z}_2}{z_3-z_2} + \frac{\dot{z}_3\dot{z}_4}{z_3-z_4}\right), \qquad \ddot{z}_4 = 2\left(\frac{\dot{z}_4\dot{z}_1}{z_4-z_1} + \frac{\dot{z}_4\dot{z}_2}{z_4-z_2} + \frac{\dot{z}_4\dot{z}_3}{z_4-z_3}\right). \end{aligned} \qquad (26)$$

If we introduce eight new dependent variables $w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$ such that

$$\begin{aligned} z_1 &= w_1, \qquad z_2 = w_2, \qquad z_3 = w_3, \qquad z_4 = w_4, \\ \dot{z}_1 &= w_5, \qquad \dot{z}_2 = w_6, \qquad \dot{z}_3 = w_7, \qquad \dot{z}_4 = w_8, \end{aligned} \qquad (27)$$

then system (26) becomes an autonomous system of eight first-order equations. If we follow the reduction method and introduce a new independent variable, say $y = w_1$, then we obtain a nonautonomous system of seven equations of first order. If we derive w_6 from the first equation of that system, w_7 from the second equation and w_8 from the third equation, i.e.

$$w_6 = w_2' w_5, \qquad w_7 = w_3' w_5, \qquad w_8 = w_4' w_5,$$

then we obtain the following single equation of first order:

$$\begin{aligned} w_5' = {} & -2w_5((w_3 - y)(w_2 - y)w_4' + (w_4 - y)(w_2 - y)w_3' \\ & + (w_3 - y)(w_4 - y)w_2')/((w_2 - y)(w_3 - y)(w_4 - y)) \end{aligned} \tag{28}$$

and the following system of three equations of second order[11]

$$\begin{aligned} u_1'' = {} & 2\big(\big(\big(u_3^2 - u_3 y - u_3' y^2 + (2u_3 + y)(u_3 - y)u_2'\big)y + ((2u_3' + 1)y - u_3)u_2^2 \\ & - ((u_3 - 2y)u_3 + (u_3' + 1)y^2)u_2 - (u_2 + u_3)(u_2 - y)(u_3 - y)u_1'\big)u_1 \\ & + ((u_2 - y)u_3' + (u_3 - y)u_2')u_1^3 + (u_2 - y)(u_3 - y)u_1' u_2 u_3 \\ & + (u_2 - y)\big(u_3^2 - u_3 y - u_3' y^2\big)u_2 - (u_3 - y)u_2' u_3 y^2 + (((2u_3' + 1)y - u_3)y \\ & - u_2^2 u_3' - (u_3 + 2y)(u_3 - y)u_2' - ((u_3' + 1)y - u_3)u_2 \\ & + (u_2 - y)(u_3 - y)u_1')u_1^2\big)u_1' / ((u_1 - u_2)(u_1 - u_3)(u_1 - y)(u_2 - y)(u_3 - y)) \\ u_2'' = {} & \big(-2\big((u_2 - u_3)(u_2 - y)^2(u_3 - y)u_1' - u_2^3 u_3' y + \big(u_2^2 u_3' + u_2 u_2' u_3 - u_2 u_2' y \\ & + u_2 u_3 - 2u_2 u_3' y - u_2 y - u_2' u_3^2 + u_2' u_3 y - u_3^2 + u_3 y + u_3' y^2\big)(u_2 + y)u_1 \\ & + \big(u_3^2 - u_3 y - u_3' y^2 + (u_3 - y)u_2' u_3\big)u_2 y + ((2u_3' + 1)y - u_3 \\ & - (u_3 - y)u_2')u_2^2 y + \big((u_3 - y)u_2' u_3 - u_2^2 u_3' + u_3^2 - u_3 y - u_3' y^2 + ((2u_3' + 1)y \\ & - u_3 - (u_3 - y)u_2')u_2\big)u_1^2\big)u_2'\big) \big/ ((u_1 - u_2)(u_1 - y)(u_2 - u_3)(u_2 - y)(u_3 - y)) \\ u_3'' = {} & \big(2\big(\big(\big(u_2^2 + u_3 y\big)(u_3' + 1) - (u_3 - y)^2 u_2' - (u_3 + y)(u_3' + 1)u_2\big)u_3 y \\ & - (u_2 - u_3)(u_2 - y)(u_3 - y)^2 u_1' + \big(((u_3 + y)u_2 - u_3 y)(u_3' + 1) + (u_3 - y)^2 u_2' \\ & - (u_3' + 1)u_2^2\big)(u_3 + y)u_1 + \big((u_2^2 + u_3 y)(u_3' + 1) - (u_3 - y)^2 u_2' \\ & - (u_3 + y)(u_3' + 1)u_2\big)u_1^2\big)u_3'\big) \big/ ((u_1 - u_3)(u_1 - y)(u_2 - u_3)(u_2 - y)(u_3 - y)) \end{aligned} \tag{29}$$

which admits a 24-dimensional Lie symmetry algebra isomorphic to $sl(5, \mathbb{R})$ [5, 6], i.e. it is linearizable [4, 19] by means of the following point transformation:

$$\begin{aligned} \tilde{y} &= y + u_1 + u_2 + u_3 = z_1 + z_2 + z_3 + z_4 \\ \tilde{u}_1 &= y u_1 u_2 u_3 = z_1 z_2 z_3 z_4, \\ \tilde{u}_2 &= y u_1 u_2 + y u_1 u_3 + y u_2 u_3 + u_1 u_2 u_3 \\ &= z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4, \\ \tilde{u}_3 &= y u_1 + y u_2 + y u_3 + u_1 u_2 + u_1 u_3 + u_2 u_3 \\ &= z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4, \end{aligned} \tag{30}$$

and system (29) becomes

$$\frac{\mathrm{d}^2 \tilde{u}_1}{\mathrm{d}\tilde{y}^2} = 0, \qquad \frac{\mathrm{d}^2 \tilde{u}_2}{\mathrm{d}\tilde{y}^2} = 0, \qquad \frac{\mathrm{d}^2 \tilde{u}_3}{\mathrm{d}\tilde{y}^2} = 0, \tag{31}$$

which may be interpreted as the equations of one-dimensional motion of three free particles.

[11] To simplify the notation we have replaced w_2 with u_2, w_3 with u_3, and w_4 with u_1. Note that equation (28) can be easily integrated as soon as the general solution of system (29) is determined.

We remark that without the help of Lie group analysis it would be quite impossible to find out that the reduced system (29), which is even more complex than the 'goldfish' system (14) in the case $N = 3$, is actually linearizable and also determine the transformation which linearizes it.

2.4. Any N

It is now clear that to transform the 'goldfish' into a school of free particles, firstly we have to reduce the N equation of second order (1) to an autonomous system of $2N$ equations of first order by introducing the new dependent variables w_k $(k = 1, \ldots, 2N)$ such that

$$z_1 = w_1, \ldots, z_N = w_N, \qquad \dot{z}_1 = w_{N+1}, \ldots, \dot{z}_N = w_{2N}. \tag{32}$$

If we follow the reduction method and introduce a new independent variable, say $y = w_1$, then we obtain the following non-autonomous system of $2N - 1$ equations of first order

$$\begin{aligned}
w_2' &= \frac{w_{N+2}}{w_{N+1}} \\
w_3' &= \frac{w_{N+3}}{w_{N+1}} \\
&\vdots \\
w_N' &= \frac{w_{2N}}{w_{N+1}} \\
w_{N+1}' &= 2\sum_{m=2}^{N} \frac{w_{N+m}}{y - w_m} \\
w_{N+2}' &= 2\left(\frac{w_{N+2}}{w_2 - y} + \sum_{m=3}^{N} \frac{w_{N+2}w_{N+m}}{w_{N+1}(w_2 - w_m)}\right) \\
w_{N+3}' &= 2\left(\frac{w_{N+3}}{w_3 - y} + \sum_{m=2, m\neq 3}^{N} \frac{w_{N+3}w_{N+m}}{w_{N+1}(w_3 - w_m)}\right) \\
&\vdots \\
w_{2N}' &= 2\left(\frac{w_{2N}}{w_N - y} + \sum_{m=2}^{N-1} \frac{w_{2N}w_{N+m}}{w_{N+1}(w_N - w_m)}\right),
\end{aligned} \tag{33}$$

where $'$ denotes derivative by y. If we derive w_{N+2} from the first equation of system (33), w_{N+3} from the second equation ..., and w_{2N} from the $(N - 1)$th equation, i.e.

$$w_{N+2} = w_2' w_{N+1}, \qquad w_{N+3} = w_3' w_{N+1}, \ldots, w_{2N} = w_N' w_{N+1},$$

then we obtain one (easy to integrate) equation of first order in w_{N+1}, and a system of $N - 1$ equations of second order in $u_j = w_{N+1+j}$ $(j = 1, \ldots, N - 1)$, which admits a Lie symmetry algebra of dimension $(N - 1)^2 + 4(N - 1) + 3 = N(N + 2)$ isomorphic to $sl(N + 1, \mathbb{R})$ [5, 6], i.e. it is linearizable [4, 19] by means of the following point transformation:

$$\begin{aligned}
\tilde{y} &= y + u_1 + u_2 + \cdots + u_{N-1} = \sum_{n=1}^{N} z_n \\
\tilde{u}_1 &= y u_1 u_2 \cdots u_{N-1} = \prod_{n=1}^{N} z_n, \\
&\vdots
\end{aligned}$$

$$\tilde{u}_{N-2} = yu_1u_2 + yu_1u_3 + \cdots + u_{N-3}u_{N-2}u_{N-1} = \sum_{\substack{n_1,n_2,n_3=1\\ n_1<n_2<n_3}}^{N} z_{n_1}z_{n_2}z_{n_3}$$

$$\tilde{u}_{N-1} = yu_1 + yu_2 + \cdots + u_{N-2}u_{N-1} = \sum_{\substack{n_1,n_2=1\\ n_1<n_2}}^{N} z_{n_1}z_{n_2}$$

and system (1) becomes (3) which may be interpreted as the equations of one-dimensional motion of $N-1$ free particles. We may also consider $\tilde{y}, \tilde{u}_j$ $(j = 1, \ldots, N-1)$ as the obvious coefficients of the following polynomial of degree N in z:

$$\prod_{n=1}^{N}(z - z_n). \tag{34}$$

Once the general solution of system (3) is trivially determined and substituted into the polynomial (34), then its roots yield the general solution of system (1).

3. Some final remarks

The purpose of this paper was to exemplify once more the power of Lie group analysis. One could know nothing about Calogero's derivation of system (1) and still be able to unveil its nice properties thanks to Lie's method. We conjecture that other solvable many-body problems could be 'framed' if the same technique that we have used in this paper is applied.

As Jacobi said in the introduction to his lectures on dynamics [8]:

> $\cdots$ jeder Fortschritt in der Theorie der partiellen Differentialgleichungen auch einen Fortschritt in der Mechanik herbeiführen muss[12].

It is worth underlining *the nature and extent of Jacobi's influence upon Lie* [7] especially as this year is Jacobi's bicentennial.

Finally, it is a remarkable happenstance that 'the mathematical mermaid', as the problem of finding a third case of integrability of the problem of a heavy rigid body with a fixed point was called [9], and Calogero's 'goldfish', can each be 'fished' by using Lie group analysis as we have shown in [12] and in this paper, respectively.

References

[1] Calogero F 1978 Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations and related 'solvable' many-body problems *Nuovo Cimento* B **43** 177–241
[2] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments* (Berlin: Springer)
[3] Calogero F 2001 The neatest many-body problem amenable to exact treatments (a 'goldfish'?) *Physica* D **152–153** 78–84
[4] Fels M E 1995 The equivalence problem for systems of second-order ordinary differential equations *Proc. Lond. Math. Soc.* III **71** 221–40
[5] González-Gascón F and González-López A 1983 Symmetries of differential equations: IV *J. Math. Phys.* **24** 2006–21
[6] González-López A 1988 Symmetries of linear systems of second-order ordinary differential equations *J. Math. Phys.* **29** 1097–105
[7] Hawkins T 1991 Jacobi and the birth of Lie's theory of groups *Arch. Hist. Exact Sciences* **42** 187–278
[8] Jacobi K G J 1886 *Vorlesungen über Dynamik* (Berlin: Druck und Verlag von Georg Reimer)

[12] Any progress in the theory of partial differential equations must also bring about a progress in mechanics.

[9] Kowalevski S 1891 An autobiographical sketch *Russkaya starina* **1** 450–63 (in Russian)
Kovalevskaya S 1978 *A Russian Childhood* (New York: Springer) (Engl. Transl.)

[10] Leach P G L and Nucci M C 2004 Reduction of the classical MICZ-Kepler problem to a two-dimensional linear isotropic harmonic oscillator *J. Math. Phys.* **45** 3590–604

[11] Lie S 1912 *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen* (Leipzig: Teubner)

[12] Marcelli M and Nucci M C 2003 Lie point symmetries and first integrals: the Kowalevsky top *J. Math. Phys.* **44** 2111–32

[13] Noether E 1918 Invariante Variationsprobleme *Kgl. Ges. Wiss. Nach. Math-phys. Kl. Heft* **2** 235–69

[14] Nucci M C 1990 Interactive REDUCE programs for calculating classical, non-classical and Lie–Bäcklund symmetries of differential equations *Georgia Tech. Math. Preprint* 062090–051

[15] Nucci M C 1996 Interactive REDUCE programs for calculating Lie point, non-classical, Lie–Bäcklund, and approximate symmetries of differential equations: manual and floppy disk (*CRC Handbook of Lie Group Analysis of Differential Equations vol 3: New Trends in Theoretical Developments and Computational Methods*) ed N H Ibragimov (Boca Raton: CRC Press) pp 415–81

[16] Nucci M C 1996 The complete Kepler group can be derived by Lie group analysis *J. Math. Phys.* **37** 1772–5

[17] Nucci M C 2003 Lorenz integrable system moves à la Poinsot *J. Math. Phys.* **44** 4107–18

[18] Nucci M C and Leach P G L 2001 The harmony in the Kepler and related problems *J. Math. Phys.* **42** 746–64

[19] Soh C W and Mahomed F M 2001 Linearization criteria for a system of second-order ordinary differential equations *Int. J. Non-linear Mech.* **36** 671–7

[20] Torrisi V and Nucci M C 2001 Application of Lie group analysis to a mathematical model which describes HIV transmission *The Geometrical Study of Differential Equations* ed J A Leslie and T P Hobart (Providence, RI: AMS) pp 31–40

[21] Zakharov V E 1990 On the dressing method *Inverse Methods in Action* ed P C Sabatier (Berlin: Springer) pp 602–23 (see p 622)